Connection between the Burgers equation with an elastic forcing term and a stochastic process

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In this paper, a complete analytical resolution of the one dimensional Burgers equation with the elastic forcing term $-\kappa^2 x + f(t)$, $\kappa \in \mathbb{R}$ is presented. Two methods existing for the case $\kappa = 0$ are adapted and generalized using variable and functional transformations, valid for all values of space and time. The emergence of a Fokker-Planck equation in the method allows the establishment of a connection between the Burgers equation and the Ornstein-Uhlenbeck process.

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I. INTRODUCTION

The Burgers equation is known to have a lot in common with the Navier-Stokes equation. In particular, it presents the same kind of advective nonlinearity, and a Reynolds number may be defined from the viscosity term [1]. This nonlinear equation is also frequently used as a model for statistical theories of turbulence and shock waves, from which asymptotical behaviors may be determined in the limit of vanishing viscosity [2,3]. The numerous applications of this equation have led to focus on statistical behavior of solutions, in particular in the case of the forced Burgers equation [4–6]. The Burgers equation may also appear in magnetohydrodynamics, where the resolution presents additional difficulties due to the nonlinear coupling between the Burgers equation and the Maxwell equations [7,8]. Thus, from an analytical point of view, the inhomogeneous version of the Burgers equation is little studied, the complete analytical solution being closely dependent of the form of the forcing term. Furthermore, while the solution of the one-dimensional homogeneous Burgers equation is well known (for a multidimensional resolution see Ref. [9]), it is advisable to remind briefly the integrable case of the following noisy Burgers equation (inhomogeneous Burgers equation with a time-dependent forcing term), which has been in the focus of quite a number of studies [10–15]

$$\begin{cases} \partial_t u + u\partial_x u - \nu\partial_{xx} u = f(t), \\ u(x,0) = \varphi(x). \end{cases} \tag{1}$$

It appears that the solution in this case may be obtained by two methods. The first method lies on the Orlowsky-Sobczyk (OS) transformations [16], where the inhomogeneous Burgers equation (1) is transformed into a homogeneous Burgers equation. Nevertheless, another equivalent method may be used to solve this problem analytically. Using the well-known Hopf-Cole transformation [17], an inhomogeneous Burgers equation may be transformed into a linear equation: the heat equation with a source term, which may be compared to a Schrödinger equation with an imaginary time, and a space-and-time dependent potential. Several methods have been developed over past decades to treat this kind of equation. One of them, the time-space transformation (TST) method, has been used in order to solve the Schrödinger equation with a time-dependent mass moving in a time-dependent linear potential (Feng [18]). It is thus shown in Ref [19]., the equivalence between the TST method and the Orlowsky-Sobczyk method, that is to say, the possibility to solve analytically by two equivalent ways, the Burgers equation with a forcing term in $f(t)$. The following diagram shows this equivalence, where Heat-S designates the heat equation with a source term, BE the Burgers equation, and HC the Hopf-Cole transformation.

$$\begin{array}{ccc} \text{Inhomogeneous BE: } f(t) & \xrightarrow{\text{OS}} & \text{Homogeneous BE} \\ \text{HC}\big\downarrow & & \big\downarrow\text{HC} \\ \text{Heat} - \text{S(linear)} & \xrightarrow{\text{TST}} & \text{Heat} \end{array}$$

This yields to presenting this paper as a continuation of the previous existing methods. The two latest methods (OS and TST) are adapted in order to solve the inhomogeneous Burgers equation with a forcing term of the form $-\kappa^2 x + f(t)$, where the value κ^2 represents the string constant of the elastic force. Let us note that Wospakrik and Zen [20] have treated this problem, but only in the limiting case of vanishing viscosity for the asymptotic mode, whereas the methods presented here are valid in all cases. The outline of the paper will be thus as follows: Sec. II will be devoted to the treatment of an elastic term, first by the way of a TST method, and then by using a generalized OS method. It is then shown that a Fokker-Planck equation, associated to the Ornstein-Uhlenbeck process, arises in the resolution by the TST method. Consequently, an "adapted" Hopf-Cole transformation may be obtained for this case, and a physical interpretation in the asymptotic limit is discussed. The connection between the Burgers equation and the Ornstein-Uhlenbeck process is in keeping with stochastic diffusion processes described by Burgers' dynamic (see e.g., Ref. [21].).

II. RESOLUTION FOR AN ELASTIC FORCING TERM

As underlined in the introduction, the TST method allows us to solve a Schrödinger equation for some kinds of potentials. So the inhomogeneous Burgers equation has firstly to be transformed into such an equation. Starting from the following one-dimensional Burgers equation with a linear forcing term

$$\begin{cases} \partial_t u + u\partial_x u - \nu\partial_{xx} u = -\kappa^2 x + f(t), \\ u(x,0) = \varphi(x), \end{cases} \tag{2}$$

we apply a Hopf-Cole transformation of the form $u(x,t) = -2\nu[1/\Psi(x,t)]\partial_x\Psi(x,t)$ to obtain a heat equation with a source term S

$$\partial_t\Psi(x,t) = \nu\partial_{xx}\Psi(x,t) + S(x,t)\Psi, \tag{3}$$

where $S(x,t) = (\kappa^2/4\nu)x^2 - [f(t)/2\nu]x + c(t)$, $c(t)$ being an arbitrary time-dependent function. This kind of equation allows the application of a TST method based on several changes of variables. In Ref. [19], and following Ref. [18], a TST method has been used in order to solve a Schrödinger equation with a linear potential. Here, a quadratic potential appears in Eq. (3), so the method will consist this time to put

$$\Psi(x,t) = P(x,t)e^{h(x,t)}, \tag{4}$$

with $h(x,t) = a_1x^2 + a_2(t)x + a_3(t)$; a_1, $a_2(t)$, and $a_3(t)$ being constant or time-dependent functions to be determined. The transformation (4) introduced in Eq. (3) gives

$$\partial_t P = \nu\partial_{xx}P + 2\nu\partial_x h\partial_x P + P[\nu\partial_{xx}h + \nu(\partial_x h)^2 + S - \partial_t h]. \tag{5}$$

Then, in order to cancel the factor of P, we put

$$\nu\partial_{xx}h + \nu(\partial_x h)^2 + S - \partial_t h = 0, \tag{6}$$

which gives a polynomial of second degree in x. This permits us to obtain, respectively, the following relations:

$$4\nu a_1^2 + \frac{\kappa^2}{4\nu} = 0, \tag{7a}$$

$$4\nu a_1 a_2 - \frac{f}{2\nu} - \dot{a}_2 = 0, \tag{7b}$$

$$2\nu a_1 + \nu a_2^2 + c - \dot{a}_3 = 0. \tag{7c}$$

Since Eqs. (7) are satisfied, Eq. (5) is simplified to

$$\partial_t P = \nu\partial_{xx}P + 2\nu\partial_x h\partial_x P. \tag{8}$$

We now apply to Eq. (8) the following change of variables

$$\begin{cases} y = r(t)x + q(t), \\ t' = t, \end{cases} \tag{9}$$

which induces a transformation of Eq. (8) into

$$\partial_{t'}P = \nu r^2\partial_{yy}P + [(-\dot{r}/r + 4\nu a_1)(y-q) + 2\nu r a_2 - \dot{q}]\partial_y P. \tag{10}$$

The simplification of this equation is made by putting

$$\dot{r} - 4\nu a_1 r = 0, \tag{11a}$$

$$2\nu r a_2 - \dot{q} = 0. \tag{11b}$$

Notice that the relation (7a) gives

$$a_1 = i\frac{\kappa}{4\nu}, \tag{12}$$

where $i = \sqrt{-1}$, with the result that the solution of Eq. (11a) will be

$$r(t) = e^{i\kappa t}. \tag{13}$$

Equation (11) being satisfied, we obtain

$$\partial_{t'}P = \nu r^2\partial_{yy}P. \tag{14}$$

Finally, the transformation

$$\tau(t') = \int_0^{t'} r^2(s)ds, \tag{15}$$

yields to the expected heat equation

$$\partial_\tau P(y,\tau) = \nu\partial_{yy}P(y,\tau). \tag{16}$$

We will now show that the Orlowsky-Sobczyk method is a particular case of the method employed here for an elastic term: the generalized Orlowsky-Sobczyk (GOS) method. Let us consider again Eq. (2), and let us introduce a new velocity $v \equiv v(x,t)$ such as

$$u = vr(t) + \alpha x + \psi(t), \tag{17}$$

where $r(t), \alpha, \psi(t)$ are time-dependent functions or constants determined later. The transformation (17) introduced in Eq. (2) yields to

$$v[\dot{r} + \alpha r] + x[\kappa^2 + \alpha^2] + [\dot{\psi} + \alpha\psi - f] + r\partial_t v + r^2 v\partial_x v + \alpha r x\partial_x v + r\psi\partial_x v - \nu r\partial_{xx}v = 0. \tag{18}$$

In order to cancel the terms in v and x, and those only depending on time, we put

$$\dot{r} + \alpha r = 0, \tag{19a}$$

$$\kappa^2 + \alpha^2 = 0, \tag{19b}$$

$$\dot{\psi} + \alpha\psi - f = 0. \tag{19c}$$

Since the system (19) is verified, then Eq. (18) is simplified into

$$r\partial_t v + r^2 v\partial_x v + \alpha r x\partial_x v + r\psi\partial_x v - \nu r\partial_{xx}v = 0. \tag{20}$$

Then, the same time and space change of variables as Eq. (9) applied to Eq. (20) leads to

$$p\partial_{t'}v + \partial_y v[r\dot{q} + r^2\psi] + (y-q)\partial_y v[\dot{r} + \alpha r] + r^3 v\partial_y v - \nu r^3\partial_{yy}v = 0. \tag{21}$$

Putting then

$$r\dot{q} + r^2\psi = 0 \tag{22}$$

we obtain

$$\frac{1}{r^2}\partial_{t'}v + v\partial_y v = \nu\partial_{yy}v. \tag{23}$$

So the last transformation

$$\tau(t') = \int_0^{t'} r^2(s)ds, \tag{24}$$

gives rise to a homogeneous Burgers equation governing the new velocity v

$$\partial_\tau v + v\partial_y v = \nu \partial_{yy} v. \tag{25}$$

From this, the HC transformation $v=-2\nu(1/P)\partial_y P$ yields again to the expected heat equation

$$\partial_\tau P(y,\tau) = \nu \partial_{yy} P(y,\tau). \tag{26}$$

Hence, both methods GOS and TST may be connected thanks to a commutative diagram similar to the one of the introduction, with a force $-\kappa^2 x + f(t)$.

III. DERIVATION OF AN ORNSTEIN-UHLENBECK PROCESS

Let $x(t)$ be a stochastic variable satisfying the following Langevin equation and describing an Ornstein-Uhlenbeck process [22,23]:

$$\frac{dx}{dt} = -\kappa x + \sqrt{2\nu}\, b(t), \tag{27}$$

where $b(t)$ stands for a Gaussian white noise verifying the standard conditions

$$\langle b(t)\rangle = 0 \text{ and } \langle b(t)b(t')\rangle = \delta(t-t'). \tag{28}$$

Then, using a Kramers-Moyal expansion, a Fokker-Planck equation may be obtained for the transition probability $P(x,t)$ [24]

$$\partial_t P(x,t) = \kappa \partial_x [xP(x,t)] + \nu \partial_{xx} P(x,t). \tag{29}$$

This equation is usually solved by Fourier transform, and the solution $P \equiv P(x,x',t)$ for the initial condition $P(x,t|x',0) = \delta(x-x')$ reads

$$P = \sqrt{\frac{\kappa}{2\pi\nu(1-e^{-2\kappa t})}} \exp\left[-\frac{\kappa(x-e^{-\kappa t}x')^2}{2\nu(1-e^{-2\kappa t})}\right]. \tag{30}$$

It is shown in appendix that this solution may also be found by the TST method. The interesting point lies in a connection between the Ornstein-Uhlenbeck process [Eq. (29)] and the Burgers equation (2) with $f(t)=0$. In order to show this fact, we apply the transformation

$$P(x,t) = \Psi(x,t) e^{-(\kappa x^2/4\nu)}, \tag{31}$$

to the Fokker-Planck equation (29). This leads to the heat equation

$$\partial_t \Psi = \nu \partial_{xx}\Psi + \left(\frac{\kappa}{2} - \frac{\kappa^2 x^2}{4\nu}\right)\Psi. \tag{32}$$

So, the Hopf-Cole transformation

$$u(x,t) = -2\nu \frac{1}{\Psi(x,t)} \partial_x \Psi(x,t), \tag{33}$$

transforms Eq. (32) into the inhomogeneous Burgers equation

$$\partial_t u + u\partial_x u = \nu\partial_{xx} u - \kappa^2 x. \tag{34}$$

From this result, the following consequences may be drawn. This connection gives rise to a physical meaning of the TST method. Indeed, the function P introduced in the transformation (4) takes the sense of a transition probability for the variable $x(t)$. Then, considering both Eqs. (31) and (33), we obtain a relation between the velocity u and the transition probability P

$$u(x,t) = -2\nu \frac{1}{P(x,t)} \partial_x P(x,t) - \kappa x, \tag{35}$$

which is composed of a Hopf-Cole part and of a linear part. Hence, this relation may be considered as a Hopf-Cole transformation adapted to the Ornstein-Uhlenbeck process. Moreover, the asymptotic limit of $P(x,x',t)$ is given by (30)

$$\lim_{t\to\infty} P(x,x',t) = \sqrt{\frac{\kappa}{2\pi\nu}} \exp\left(-\frac{\kappa x^2}{2\nu}\right), \tag{36}$$

and thus, from the relation (35), it can be seen that the asymptotic limit of the velocity will read

$$\lim_{t\to\infty} u(x,t) = \kappa x, \tag{37}$$

which is a stationary solution. Thus, the velocity associated to the Ornstein-Uhlenbeck process behaves asymptotically linear with x. This result being obtained is thanks to the initial condition $P(x,t|x',0)=\delta(x-x')$, which expresses the necessary condition that, for example, a particle cannot be initially at several positions. In other words, any initial condition of the form $P(x,t|x',0)=C\delta(x-x'), C\in\mathbb{R}^+$, has a physical meaning. We can, therefore, notice from (35), that all theses initial conditions yield to the same result for the value $\lim_{t\to\infty} u(x,t)$. It follows that the relation (37) may be considered valid whatever the initial condition on the velocity may be. It can be concluded that an elastic forcing term applied to the system gives rise to a stationary transition probability in the asymptotic mode. Consequently, the effects of the oscillations will decrease, even disappearing in the long-time limit, and stabilize the system with a velocity proportional to the displacement. The evanescence of the effect of the force is due to the initial condition sensitivity of the Burgers equation. We can see thereby on the system, a phenomenon closely connected to the turbulence effect: the loss of memory in the long-time limit.

IV. CONCLUSION

In this paper, we have presented the complete analytical solution of the Burgers equation with an elastic forcing term. The methods presented here have been used before but only in the case of a time-dependent forcing term. For perspective, we can say that the generalization of the methods to any order of power of x seems actually be a difficult task. Indeed, a transformation of the form $y \to r(t)x + q(t)$, has been introduced in order to delete terms proportional to x. So this transformation seems without effect when higher powers of x appear. Moreover, the higher the degree, the more difficult the resolution, due to the increasing number of variables to be introduced. The second main result of the paper lies in the existence of links between a fluid model (Burgers) and the statistical physics (Ornstein-Uhlenbeck). Through a set of transformations, we have etablished a connection between the Burgers equation for the velocity $u=dx/dt$ and a Fokker-Planck equation for the transition probability of the variable x. From the Burgers equation (34), the transformation (35) allows us to obtain directly the Fokker-Planck equation (29) as a specific Hopf-Cole transformation. It appears that the linear force, describing the Ornstein-Uhlenbeck process, stabilizes the system in the asymptotic mode with a velocity proportional to the force applied initially, whatever the initial condition on the velocity may be. This result shows a characteristic property of turbulence, i.e., the unpredictability of a velocity field governed by the Burgers equation.

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APPENDIX: SOLUTION OF THE ORNSTEIN-UHLENBECK PROCESS

We show that we can recover the solution (30) using our TST method. Rewriting Eq. (29),

$$\partial_t P = \nu \partial_{xx} P + \kappa x \partial_x P + \kappa P, \tag{A1}$$

we apply the change of variable

$$\begin{cases} y = r(t)x, \\ t' = t. \end{cases} \tag{A2}$$

This yields to

$$\partial_{t'} P = \nu r^2 \partial_{yy} P + \left(\kappa - \frac{\dot{r}}{r}\right) y \partial_y P + \kappa P. \tag{A3}$$

To cancel the term in $\partial_y P$, we put obviously

$$\kappa - \frac{\dot{r}}{r} = 0 \Leftrightarrow r(t') = e^{\kappa t'}. \tag{A4}$$

This leads to

$$\partial_{t'} P = \nu r^2 \partial_{yy} P + \kappa P. \tag{A5}$$

Then, putting

$$P(y,t') = \Theta(y,t') e^{\kappa t'}, \tag{A6}$$

followed by the transformation

$$\tau(t') = \int_0^{t'} r^2(s) ds, \tag{A7}$$

we obtain the heat equation

$$\partial_\tau \Theta = \nu \partial_{yy} \Theta. \tag{A8}$$

Notice that the condition $P(y,y',0)=\delta(y-y')$ implies that $\Theta(y,y',0)=\delta(y-y')$. The fundamental solution of (A8) is thus

$$\Theta(y,\tau) = \frac{1}{\sqrt{4\pi\nu\tau}} \exp\left[-\frac{(y-y')^2}{4\nu\tau}\right], \tag{A9}$$

after which, putting y and τ in place of their expression, it is to say

$$\begin{cases} y = xe^{\kappa t}, \\ \tau = \dfrac{1}{2\kappa}(e^{2\kappa t'} - 1), \end{cases} \tag{A10}$$

we obtain

$$P = \sqrt{\frac{\kappa}{2\pi\nu(1-e^{-2\kappa t})}} \exp\left[-\frac{\kappa(x - e^{-\kappa t}x')^2}{2\nu(1-e^{-2\kappa t})}\right], \tag{A11}$$

which is the same result as Eq. (30).

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